An Improved Modal Expansion Method for Two Cascaded Junctions and Its Application to Waveguide Filters

Zhongxiang Shen and Robert H. MacPhie

Abstract—**An improved modal expansion method is described for the scattering matrix of a cascaded network which consists of an enlargement junction combined with a reduction junction. The new method can significantly reduce computation time and requirement of computer memory. Application to iris coupled waveguide cavity filters is demonstrated. Numerical results for a rectangular waveguide cavity filter are given.**

I. Introduction

The problem of electromagnetic scattering at waveguide discontinuities has been extensively studied for several decades. Of the many methods developed for this problem, the most rigorous is the modal expansion method [1]–[3]. In order to analyze multistep waveguide discontinuities by the modal expansion method, the generalized scattering matrix technique (GSMT) [4] is often employed to obtain the overall scattering matrix of the whole circuit. Due to the fact that the GSMT is computationally time-consuming, some improvements have been presented. Omar and Schunemann [5] demonstrated that the transmission matrix representation of waveguide discontinuities is superior to the scattering matrix representation. However, as shown by Mansour and MacPhie [6], the transmission matrix formulation requires an equal number of modes to be retained in any of the guides forming the discontinuities, which may result in incorrect numerical solutions. An improved transmission matrix formulation was then presented in [6], but it becomes inaccurate when two abrupt discontinuities (one enlargement and the other reduction) are in close proximity.

This paper presents an alternative, improved modal expansion method for cascaded waveguide junctions. The basic idea is to consider the two junctions at once by the modal expansion method [1]–[3], which directly yields the scattering matrix of the whole cascaded network. This improved scattering matrix formulation is formally exact and completely eliminates the numerical overflow problem [5] from which the transmission matrix formulation suffers and can avoid the inversion of two large matrices. Considerations of the properties of the junctions at the incident and transmitted ports of a practical filter lead to a matrix-partitioning technique to greatly reduce computation time and memory space requirement. Numerical tests show that the technique is very computationally effective and stable.

II. Improved Modal Expansion Method

Fig. 1 shows the structure of the scattering problem considered in this section; there are two waveguide junctions (one is enlargement and the other is reduction). The length of the sandwiched larger waveguide (guide 2 in Fig. 1) is assumed to be l. The traditional approach is to separately analyze these two individual junctions (A and B in Fig. 1) and then to obtain the overall scattering matrix of the cascaded network by the GSMT [4]. Instead, one can consider these two junctions at once. As shown in Fig. 1, a^+ and a^- are the incident and reflected modal amplitude column vectors in guide 1 at Junction A, and similarly c^+ and c^- are the incident and reflected modal amplitude vectors in guide 3 at Junction B, and b^+, b^-, d^+, and d^- are similarly defined in guide 2 for junctions A and B. Application of the boundary conditions for tangential electric and magnetic fields at the interface of Junction A yields [3], [7]

$$b^+ + b^- = \boldsymbol{M}_1(a^+ + a^-) \tag{1}$$

$$\boldsymbol{Y}_1(a^+ - a^-) = \boldsymbol{M}_1^T\boldsymbol{Y}_2(b^- - b^+) \tag{2}$$

where $\boldsymbol{Y}_i$, for $i = 1$, 2, and 3, is the modal admittance matrix for the ith waveguide. The superscript T denotes the transpose operation and $\boldsymbol{M}_1$ is the E-field mode-matching matrix of Junction A [7], whose size is of N_2 by N_1, where N_1 and N_2 are the numbers of modes considered in guide 1 and guide 2, respectively. Similar operation for Junction B leads to

$$d^+ + d^- = \boldsymbol{M}_2(c^+ + c^-) \tag{3}$$

$$\boldsymbol{Y}_3(c^- - c^+) = \boldsymbol{M}_2^T\boldsymbol{Y}_2(d^+ - d^-) \tag{4}$$

where $\boldsymbol{M}_2$ is the E-field mode-matching matrix of Junction B, whose size is of N_2 by N_3, where N_3 is the number of modes considered in guide 3. From Fig. 1, we know $b^+ = \boldsymbol{L}d^-$ and $d^+ = \boldsymbol{L}b^-$, where $\boldsymbol{L}$ is the diagonal transmission matrix of the sandwiched larger waveguide with $L_{n,n} = \exp(-j\beta_{2,n}l)$ as its nth diagonal element. Here $\beta_{2,n}$ is the propagation constant of the nth mode in guide 2. Substituting these relations into (1)–(4), and eliminating b^+, b^-, d^+, and d^-, one can obtain

$$\boldsymbol{Y}_1(a^+ - a^-) = -\boldsymbol{Y}_p(a^+ + a^-) + \boldsymbol{Y}_q(c^- + c^+) \tag{5}$$

$$\boldsymbol{Y}_3(c^- - c^+) = -\boldsymbol{Y}_q^T(a^+ + a^-) + \boldsymbol{Y}_s(c^- + c^+) \tag{6}$$

where

$$\boldsymbol{Y}_p = \boldsymbol{M}_1^T\boldsymbol{D}_1\boldsymbol{M}_1,\ \boldsymbol{Y}_q = \boldsymbol{M}_1^T\boldsymbol{D}_2\boldsymbol{M}_2,\ \boldsymbol{Y}_s = \boldsymbol{M}_2^T\boldsymbol{D}_1\boldsymbol{M}_2 \tag{7}$$

$$\boldsymbol{D}_1 = \boldsymbol{Y}_2(\boldsymbol{I} + \boldsymbol{L}^2)(\boldsymbol{L}^2 - \boldsymbol{I})^{-1},\ \boldsymbol{D}_2 = 2\boldsymbol{Y}_2\boldsymbol{L}(\boldsymbol{L}^2 - \boldsymbol{I})^{-1}. \tag{8}$$

$\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are diagonal matrices with imaginary elements. Therefore, the right-hand sides of (7) may be put into single summation form. Matrices $\boldsymbol{Y}_p$ and $\boldsymbol{Y}_s$ are symmetric. From (5) and (6), we can derive the overall scattering matrix of the two cascaded junctions as follows

$$\boldsymbol{S}_{11} = 2[\boldsymbol{Y}_1 - \boldsymbol{Y}_p + \boldsymbol{Y}_q(\boldsymbol{Y}_s - \boldsymbol{Y}_3)^{-1}\boldsymbol{Y}_q^T]^{-1}\boldsymbol{Y}_1 - \boldsymbol{I} \tag{9}$$

$$\boldsymbol{S}_{21} = (\boldsymbol{Y}_s - \boldsymbol{Y}_3)^{-1}\boldsymbol{Y}_q^T(\boldsymbol{I} + \boldsymbol{S}_{11}) \tag{10}$$

$$\boldsymbol{S}_{12} = \boldsymbol{Y}_1^{-1}\boldsymbol{S}_{21}^T\boldsymbol{Y}_3 \tag{11}$$

$$\boldsymbol{S}_{22} = (\boldsymbol{Y}_s - \boldsymbol{Y}_3)^{-1}[\boldsymbol{Y}_q^T\boldsymbol{S}_{12} - 2\boldsymbol{Y}_3] - \boldsymbol{I}. \tag{12}$$

It is noted that only two small matrices of sizes N_1 by N_1 and N_3 by N_3 need to be inverted. Compared with the traditional GSMT [4] for the problem shown in Fig. 1, we can see that this not only reduces the number of matrix inversions (from inverting four matrices to inverting two matrices), but also avoids the inversion of two large matrices of size N_2 by N_2. In some cases when the ratios of areas of guide 2 to that of guide 1 and guide 2 to guide 3 are very large [7], [8], we should take N_2 to be very large to get a convergent solution, as will be discussed later.

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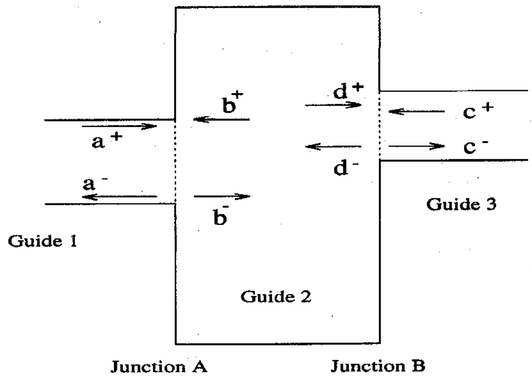

Fig. 1. Cascaded network of two waveguide junctions.

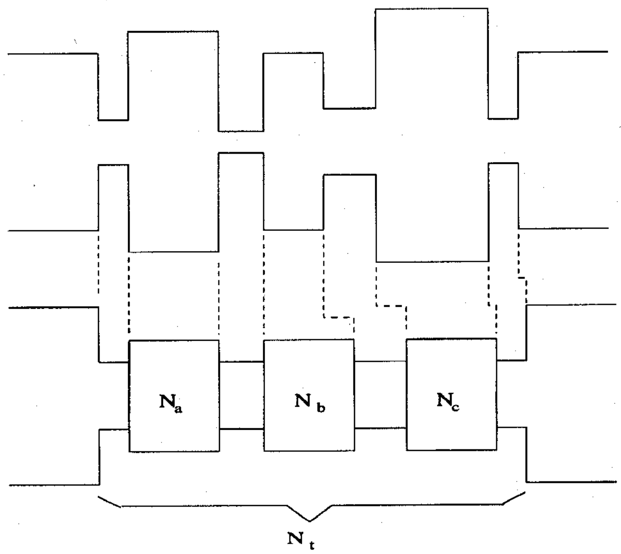

Fig. 2. Side view of a multicavity waveguide filter and its equivalent network.

III. Application to Waveguide Cavity Filters

As an illustrative example, this section considers the analysis of waveguide cavity filters [8], [9]. The side view of a waveguide cavity filter is shown in Fig. 2, which also gives its equivalent network. The overall scattering matrix of these cascaded cavities $\boldsymbol{S}^t$ can be obtained by using the GSMT [4]. Because the numbers of modes considered in the iris waveguides, say N_1 and N_3, are often small (less than or much less than 20 for most cases), all matrices which are used to obtain $\boldsymbol{S}^t$ are small. Therefore, $\boldsymbol{S}^t$ can be computed very rapidly. The effect of the thicknesses of these coupling irises can be easily taken into account with diagonal transmission matrices. After obtaining the matrix $\boldsymbol{S}^t$ for the cascaded network N_t (see Fig. 2), the problem reduces to that shown in Fig. 3, where the modal amplitude vectors of the filter input and output iris waveguide fields a^+, a^-, c^+, and c^- are related to each other in terms of the matrix S^t as follows

$$\begin{bmatrix} a^- \\ c^- \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{11}^t & \boldsymbol{S}_{12}^t \\ \boldsymbol{S}_{21}^t & \boldsymbol{S}_{22} \end{bmatrix} \begin{bmatrix} a^+ \\ c^+ \end{bmatrix}. \tag{13}$$

Referring to Fig. 3 and applying the boundary condition that tangential electric and magnetic fields must be continuous at the interfaces of Junction 1 and Junction 2, we have [3], [7]

$$b^+ + b^- = \boldsymbol{M}_i(a^+ + a^-) \tag{14}$$

$$\boldsymbol{Y}_1(a^+ - a^-) = \boldsymbol{M}_i^T \boldsymbol{Y}_i(b^+ - b^-) \tag{15}$$

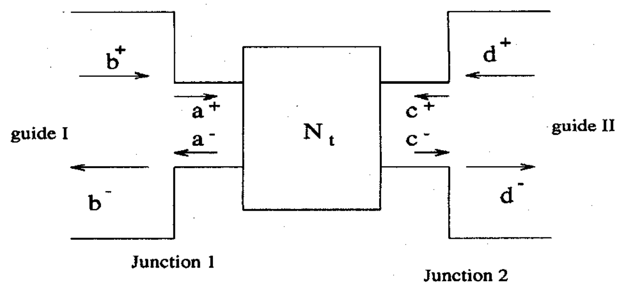

Fig. 3. Reduced structure of the waveguide cavity filter shown in Fig. 2.

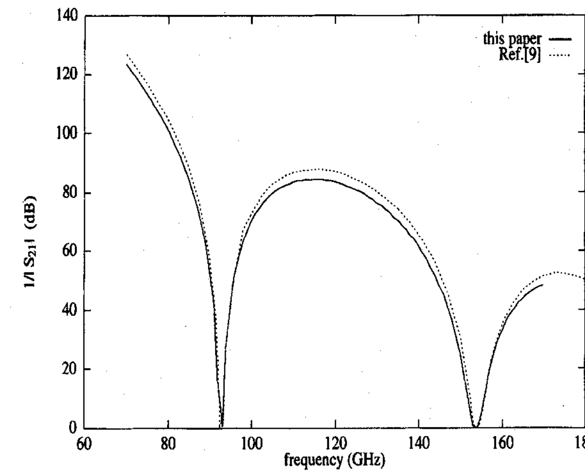

Fig. 4. Transmission response of a rectangular iris coupled three-resonator rectangular waveguide filter (a = 2.54 mm, b = 1.27 mm, $a_1 = a_4$ = 0.742 mm, $b_1 = b_4$ = 1.0937 mm, $a_2 = a_3$ = 0.4772 mm, $b_2 = b_3$ = 0.3603 mm, $l_1 = l_3$ = 1.9812 mm, l_2 = 2.0768 mm, $t_1 = t_2 = t_3 = t_4$ = 0.05 mm).

$$d^+ + d^- = \boldsymbol{M}_o(c^+ + c^-) \tag{16}$$

$$\boldsymbol{Y}_3(c^- - c^+) = \boldsymbol{M}_o^T \boldsymbol{Y}_o(d^- - d^+) \tag{17}$$

where the column vectors b^+, b^-, d^+, and d^- are the incident and reflected modal amplitude vectors of the transverse electric field components in guides I and II, respectively, which are different from those used in the previous section. Substituting (13) into (14)–(17), one may eliminate a^- and c^-. Because the incident and transmitted waveguides (guides I and II in Fig. 4) may be much larger than the iris waveguides, the sizes of vectors b and d should be very large. In almost all practical cases, we only have one or two propagating incident modes in each larger waveguide and only have interest in the reflection and transmission coefficients of these propagating modes. Accordingly, we can partition these vectors and matrices in (14)–(17) as follows

$$b^+ = \begin{bmatrix} b_l^+ \\ b_h^+ \end{bmatrix},\ b^- = \begin{bmatrix} b_l^- \\ b_h^- \end{bmatrix},\ d^+ = \begin{bmatrix} d_l^+ \\ d_h^+ \end{bmatrix},\ d^- = \begin{bmatrix} d_l^- \\ d_h^- \end{bmatrix} \tag{18}$$

$$\boldsymbol{M}_i = \begin{bmatrix} \boldsymbol{M}_{il} \\ \boldsymbol{M}_{ih} \end{bmatrix},\ \boldsymbol{M}_o = \begin{bmatrix} \boldsymbol{M}_{ol} \\ \boldsymbol{M}_{oh} \end{bmatrix},\ \boldsymbol{Y}_i = \begin{bmatrix} \boldsymbol{Y}_{il} & 0 \\ 0 & \boldsymbol{Y}_{ih} \end{bmatrix},$$

$$\boldsymbol{Y}_o = \begin{bmatrix} \boldsymbol{Y}_{ol} & 0 \\ 0 & \boldsymbol{Y}_{oh} \end{bmatrix} \tag{19}$$

where b_l^+, b_l^-, d_l^+, and d_l^- are vectors of dimension N_l for the lower-order modes in which we are interested; b_h^+, b_h^-, b_h^+, and d_h^- are for the higher-order modes. The subscripts i and o denote the input waveguide I and output waveguide II, respectively (see Fig. 3). Matrices $\boldsymbol{M}_i$, $\boldsymbol{M}_o$, $\boldsymbol{Y}_i$, and $\boldsymbol{Y}_o$ are similarly partitioned.

TABLE I
RELATIVE CONVERGENCE OF THE NORMALIZED SUSCEPTANCE OF A RECTANGULAR-TO-RECTANGULAR WAVEGUIDE JUNCTION ($a = 22.86$ mm, $b = 10.16$ mm, $f = 11$ GHz)

N_2	Case 1	Case 2	Case 3
20	2.869	18.221	109.196
42	2.918	18.657	119.175
72	2.955	18.917	120.137
110	2.978	19.189	122.094
156	2.984	19.239	123.863
210	2.989	19.465	124.269
272		19.518	125.344
342		19.563	126.337
506			126.968
600			127.761
702			128.123
812			128.283

Case 1: $a_1 = 11.43mm$, $b_1 = 5.08mm$;
Case 2: $a_1 = 6.86mm$, $b_1 = 3.16mm$;
Case 3: $a_1 = 3.86mm$, $b_1 = 1.66mm$.

In most cases, $N_l = 1$ or 2, and $b_h^+ = 0$, $d_h^+ = 0$. Furthermore, we are only interested in b_l^- and d_l^-, that is to say, we can eliminate b_h^- and d_h^- in the above equations. Then we can derive the scattering matrix of the whole filter network for the first N_l modes

$$\boldsymbol{S}_{11}^w = \boldsymbol{M}_{il}(\boldsymbol{I} + \boldsymbol{S}_{11}^t + \boldsymbol{S}_{12}^t\boldsymbol{Y}_t)\boldsymbol{Y}_u - \boldsymbol{I} \quad (20)$$

$$\boldsymbol{S}_{21}^w = \boldsymbol{M}_{ol}[\boldsymbol{S}_{21}^t + (\boldsymbol{I} + \boldsymbol{S}_{22}^t)\boldsymbol{Y}_t]\boldsymbol{Y}_u \quad (21)$$

$$\boldsymbol{S}_{12}^w = \boldsymbol{M}_{il}[\boldsymbol{S}_{12}^t + (\boldsymbol{I} + \boldsymbol{S}_{11}^t)\boldsymbol{Y}_v]\boldsymbol{Y}_w \quad (22)$$

$$\boldsymbol{S}_{22}^w = \boldsymbol{M}_{ol}(\boldsymbol{I} + \boldsymbol{S}_{22}^t + \boldsymbol{S}_{21}^t\boldsymbol{Y}_v)\boldsymbol{Y}_w - \boldsymbol{I} \quad (23)$$

where

$$\boldsymbol{Y}_t = [\boldsymbol{Y}_3 + \boldsymbol{Y}_M - (\boldsymbol{Y}_3 - \boldsymbol{Y}_M)\boldsymbol{S}_{22}^t]^{-1}(\boldsymbol{Y}_3 - \boldsymbol{Y}_M)\boldsymbol{S}_{21}^t \quad (24)$$

$$\boldsymbol{Y}_u = 2[\boldsymbol{Y}_1 + \boldsymbol{Y}_L + (\boldsymbol{Y}_L - \boldsymbol{Y}_1)(\boldsymbol{S}_{11}^t + \boldsymbol{S}_{12}^t\boldsymbol{Y}_t)]^{-1}\boldsymbol{M}_{il}^T\boldsymbol{Y}_{il} \quad (25)$$

$$\boldsymbol{Y}_v = [\boldsymbol{Y}_1 + \boldsymbol{Y}_L - (\boldsymbol{Y}_1 - \boldsymbol{Y}_L)\boldsymbol{S}_{11}^t]^{-1}(\boldsymbol{Y}_1 - \boldsymbol{Y}_L)\boldsymbol{S}_{12}^t \quad (26)$$

$$\boldsymbol{Y}_w = 2[\boldsymbol{Y}_3 + \boldsymbol{Y}_M + (\boldsymbol{Y}_M - \boldsymbol{Y}_3)(\boldsymbol{S}_{22}^t + \boldsymbol{S}_{21}^t\boldsymbol{Y}_v)]^{-1}\boldsymbol{M}_{ol}^T\boldsymbol{Y}_{ol} \quad (27)$$

with $\boldsymbol{Y}_L = \boldsymbol{M}_i^T\boldsymbol{Y}_i\boldsymbol{M}_i$ and $\boldsymbol{Y}_M = \boldsymbol{M}_o^T\boldsymbol{Y}_o\boldsymbol{M}_o$. We only need to invert four small matrices to obtain the overall scattering matrix. Using the traditional GSMT [4], we must invert six matrices for the same results. The partitioning technique also reduces the number of matrix multiplications and computer memory space (some matrices for the traditional GSMT may be very large).

IV. NUMERICAL RESULTS

We consider a double-plane step rectangular waveguide filter, which has been studied by Papziner and Arndt [9]. Only the case of air-filled waveguides and an incident TE_{10} mode are assumed. First, we study the relative convergence of a double-plane step rectangular waveguide junction for different sizes of the smaller waveguide. Table I gives numerical results for the normalized susceptance of the rectangular-to-rectangular waveguide junction $\overline{B} = j(1 - \rho_{10})/(1 + \rho_{10})$, where ρ_{10} is the reflection coefficient of the dominant TE_{10} mode. When the smaller waveguide is cut off, the normalized susceptance is purely real. From Table I, we can see that when the smaller waveguide becomes very small, the convergence is very slow, and we should retain many more modes in the larger waveguide. For the results presented in Table I, the number of modes in the smaller waveguide is fixed at 20 (12 TE modes and 8 TM modes).

TABLE II
COMPARISON OF COMPUTATION TIME OF OUR IMPROVED FORMULATION AND THE TRADITIONAL GENERALIZED SCATTERING MATRIX TECHNIQUE [4] ($a = 22.86$ mm, $b = 10.16$ mm, $f = 11$ GHz, $d1 = 15$ mm, $t1 = t2 = 0.1$ mm, $a_1 = a_2 = 3.86$ mm, $b_1 = b_2 = 1.66$ mm)

Number of modes considered		Computation time (seconds)	
N_1 for iris guide	N_2 for guide I	Our method	GSMT
20	20	0.036	0.060
20	42	0.042	0.138
20	72	0.057	0.379
20	110	0.064	1.006
20	156	0.079	3.062
20	210	0.097	9.210
20	272	0.111	20.34
20	342	0.141	
20	420	0.164	
20	930	0.382	
20	1640	0.622	

As for the problem of the waveguide cavity filter, we should also take many more modes of the larger waveguide into account when the coupling irises become very small. To compare our improved modal expansion method with the traditional GSMT, we consider a waveguide filter consisting of only one rectangular cavity. The smaller the size of the iris waveguide, the larger will be the number of modes in the larger waveguide (as shown in Table I). We may fix the size of the iris waveguides and compare the computation time for different modes considered in the larger waveguide. Table II shows the comparison of computation times for a single-resonator rectangular waveguide filter by our new formulation and the traditional GSMT. The waveguide is standard WR90 (X-band) and the central rectangular iris waveguide has 2.76% of its cross-sectional area. When the number N_2 of modes considered in the larger waveguide increases, the computation time of the traditional GSMT increases dramatically since the number of multiplication operations involved in LU factorization of a matrix is proportional to the cube of the size of the matrix. For our improved method, however, the computation time increase is less than linear with respect to N_2. Moreover, there is a great reduction in computer memory requirements. For the traditional GSMT, it was impossible with an HP series 735 workstation computer to carry out the calculations when N_2 was more than 300. Our improved method easily treated more than 1 000 modes.

Fig. 4 compares our results for the transmission response of a W-band three-resonator rectangular waveguide filter with those given in [9]; the agreement is good. In our computation, the sizes of all matrices (equivalently, the number of modes considered in the iris waveguides) to be inverted are fixed to be 10 × 10 (6 TE modes and 4 TM modes).

V. CONCLUSION

This paper has provided a formally exact modal expansion method for cascaded waveguide junctions. It has been demonstrated that the improved scattering matrix formulation can reduce the number of matrices to be inverted and avoids the inversion of some large matrices. The method is computationally effective and can be implemented on a personal computer. Application of the improved modal expansion method to waveguide cavity filters is demonstrated.

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Comment on "Are Nonreciprocal Bi-Isotropic Media Forbidden Indeed?"

Akhlesh Lakhtakia and Werner S. Weiglhofer

***Abstract*—In a recent paper,[1] Sihvola has cast doubt on our claim that all bi-isotropic media must be reciprocal. We show that Sihvola's doubt has no rational basis.**

Consider the linear homogeneous bi-anisotropic medium whose constitutive $\underline{\underline{\tilde{\alpha}}}(\tau)$ relations are given as

$$\underline{\tilde{D}},(\underline{r},t) = \varepsilon_0 \underline{\tilde{E}}(\underline{r},t) + \int_{-\infty}^{\infty} \left[\varepsilon_0 \underline{\underline{\tilde{\chi}}}_c(\tau) \bullet \underline{\tilde{E}}(\underline{r},t-\tau) + \underline{\underline{\tilde{\alpha}}}(\tau) \bullet \underline{\tilde{B}}(\underline{r},t-\tau)\right] d\tau \tag{1a}$$

$$\underline{\tilde{H}},(\underline{r},t) = \varepsilon_0 \underline{\tilde{B}}(\underline{r},t) + \int_{-\infty}^{\infty} \left[\underline{\underline{\tilde{\beta}}}(\tau) \bullet \underline{\tilde{E}}(\underline{r},t-\tau) - \frac{1}{\mu_0} \underline{\underline{\tilde{\chi}}}_m(\tau) \bullet \underline{\tilde{B}}(\underline{r},t-\tau)\right] d\tau \tag{1b}$$

where $\underline{\underline{\tilde{\chi}}}_c(t)$ and $\underline{\underline{\tilde{\chi}}}_m(t)$ are the dyadic susceptibility kernels, while $\underline{\underline{\tilde{\alpha}}}(t)$ and $\underline{\underline{\tilde{\beta}}}(t)$are the magnetoelectric kernels. Covariance in conjunction with the mathematical consistency of the Maxwell postulates leads to the condition [1], [2]

$$Trace\left[\underline{\underline{\tilde{\alpha}}}(t) - \underline{\underline{\tilde{\beta}}}(t)\right] = 0. \tag{2}$$

A very important consequence of (2) is that all bi-isotropic media must be reciprocal [2].

Doubt has been cast by Dr. Ari H. Sihvola [3] on this conclusion, as well as on the validity of (2). We will show that his doubt does not have a rational basis. Following the publication of [2] (on which Sihvola solely bases his critical remarks), further evidence on the validity of (2) had been presented by us in several publications [4]–[8], which had been made available to Sihvola well before he finalized [3].

I. Mathematical Basis For (2)

Consider a discrete electric charge moving with a constant velocity under the influence of an electromagnetic field. This charge experiences the usual Coulomb force in a co-moving frame of reference. In the laboratory frame ($\underline{r}$,t), that force is nothing but the usual Lorentz force, which fact can be established from the Lorentz covariance of the Maxwell postulates. As a result, the primitive fields are $\underline{\tilde{E}}(\underline{r},t)$ and $\underline{\tilde{B}}(\underline{r},t)$; therefore, the induction fields must be $\underline{\tilde{D}}(\underline{r},t)$ and $\underline{\tilde{H}}(\underline{r},t)$. Consequently, these induction fields have to be expressed as functionals of the primitive fields in a material medium. For a linear homogeneous medium, (1a) and (1b) thus follow as the time-dependent constitutive relations in modern electromagnetic theory.

In a sourceless region, the Faraday equation

$$\nabla \times \underline{\tilde{E}}(\underline{r},t) = -\frac{\partial \underline{\tilde{B}}(\underline{r},t)}{\partial t} \tag{3}$$

contains only the primitive fields and is therefore not affected by the constitutive relations. The Ampere-Maxwell equation

$$\nabla \times \underline{\tilde{H}}(\underline{r},t) = -\frac{\partial \underline{\tilde{D}}(\underline{r},t)}{\partial t} \tag{4}$$

contains the induction fields. When (1a) and (1b) are substituted into (4) and the result is compared with (3), a redundancy emerges. This redundancy is removed by (2), which has covariance [1] as well as uniqueness [4] proofs.

Equation (2) has also been extended to nonhomogeneous media [9].

Sihvola has not been able to prove that (2) is mathematically incorrect; indeed, he has not even suggested that is so. Instead, he has chosen to present two media that can possibly refute (2). Let us now show that his physical counter-arguments are unsustainable.

II. Tellegen Medium

Many decades ago, Tellegen conceived manufacturing a nonreciprocal bi-isotropic medium by randomly dispersing inclusions of a special type in an appropriate host medium. Each inclusion would be made of a glued pair of two parallel dipoles, one magnetic and the other electric. The resulting medium would be isotropic: $\underline{\underline{\tilde{\chi}}}_0(t) = \tilde{\chi}_c(t)\underline{\underline{I}}, \underline{\underline{\tilde{\chi}}}_m(t) = \tilde{\chi}_m(t)\underline{\underline{I}}, \underline{\underline{\tilde{\alpha}}}(t) = \tilde{\alpha}(t)\underline{\underline{I}}$, and $\underline{\underline{\tilde{\beta}}}(t) = \tilde{\beta}(t)\underline{\underline{I}}$ where $\underline{\underline{I}}$ is the identity dyadic. More important, this composite medium would have $\tilde{\alpha}(t) + \tilde{\beta}(t) = 0$ and would thus violate (2), if it could be made!

No one has been able to make a sample of the Tellegen medium. Elsewhere, it has been mathematically proved that even if such a

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